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On the consistency strength of the FRP for the second uncountable cardinal

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Abstract

We show that the consistency strength of the Fodor-type Reflection Principle for the second uncountable cardinal is exactly that of a Mahlo cardinal.

Introduction

The Fodor-type Reflection Principles for various uncountable cardinals λ, denoted by $\mathrm{FRP}(\lambda)$, are introduced in [F]. We are interested in the consistency strength of $\mathrm{FRP}(\omega_2)$ in this note. Let us recall the following two reflection principles, where $S_0^2=\{\alpha<\omega_2\mid \mathrm{cf}(\alpha)=\omega\}$ and $S_1^2=\{\alpha<\omega_2\mid \mathrm{cf}(\alpha)=\omega_1\}$.

(1) For all stationary $S\subseteq[\omega_2]^{\omega}$, there exists $\gamma\in S_1^2$ such that $S\cap[\gamma]^{\omega}$ is stationary in $[\gamma]^{\omega}$.

(2) For all stationary $S\subseteq S_0^2$, there exists $\gamma\in S_1^2$ such that $S\cap\gamma$ is stationary in γ.

It is known that $\mathrm{FRP}(\omega_2)$ fits in between these two by [F]. Namely, (1) implies $\mathrm{FRP}(\omega_2)$. And $\mathrm{FRP}(\omega_2)$ implies (2). The consistency strength of (1) is that of a weakly compact cardinal by [V]. And the consistency strength of (2) is that of a Mahlo cardinal by [H-S]. We follow [S] (pp.576-581) to show that the consistency strength of $\mathrm{FRP}(\omega_2)$ is that of a Mahlo cardinal.

§1. Main Theorem

Definition. A map $\langle C_\delta\mid\delta\in E\rangle$ is a *ladder system*, if $E\subseteq S_0^2$ is stationary in ω_2 and each C_δ is a cofinal subset of δ such that the order-type of C_δ is ω. Let $\gamma\in S_1^2$. We say a sequence $\langle X_i\mid i<\omega_1\rangle$ is a *filtration* on γ, if it is continuously $\subseteq$-increasing countable subsets of γ with $\bigcup\{X_i\mid i<\omega_1\}=\gamma$.

The following is equivalent to the $\mathrm{FRP}(\omega_2)$ of [F] and we take this as our definition of $\mathrm{FRP}(\omega_2)$.

Definition. The *Fodor-type Reflection Principle* for the second uncountable cardinal, denoted by *$FRP(\omega_2)$*, holds, if for any ladder system $\langle C_\delta\mid\delta\in E\rangle$, there exists $\gamma\in S_1^2$ and a filtration $\langle X_i\mid i<\omega_1\rangle$ on γ such that $T=\{i<\omega_1\mid \sup(X_i)\in E \text{ and } C_{\sup(X_i)}\subseteq X_i\}$ is stationary in ω_1.

Definition. Let κ be a strongly inaccessible cardinal. The Levy collapse which makes $\kappa=\omega_2$ by the countable conditions is denoted by $\mathrm{Lv}(\kappa,\omega_1)$. Hence $p\in\mathrm{Lv}(\kappa,\omega_1)$, if p is a function whose domain is a countable subset of $[\omega_2,\kappa)\times\omega_1$ such that for all (ξ,i) in the domain of p, we demand $p(\xi,i)<\xi$. For $p,q\in\mathrm{Lv}(\kappa,\omega_1)$, we define $q\leq p$, if $q\supseteq p$.

Theorem. Let κ be a Mahlo cardinal and assume GCH in the ground model V. Let G_κ be any $\mathrm{Lv}(\kappa,\omega_1)$-generic filter over V. Then we have $\kappa=\omega_2$ and $(\kappa^+)^V=\omega_3$ in the generic extension $V[G_\kappa]$. Now in $V[G_\kappa]$, we may construct a $<\omega_2$-support ω_3-stage iterated forcing $\langle P^*_{\alpha^*}\mid\alpha^*\leq\omega_3\rangle$ such that for each $\alpha^*<\omega_3$, $P^*_{\alpha^*}$ is ω_2-Baire and has a dense subset of size ω_2 and that $\mathrm{FRP}(\omega_2)$ holds in the generic extensions $V[G_\kappa]^{P^*_{\omega_3}}$.

§2. An Idea of Proof

Let κ be a Mahlo cardinal and assume GCH in the ground model V. Let G_κ be a fixed $\mathrm{Lv}(\kappa,\omega_1)$-generic filter over V. We work in the generic extension $V[G_\kappa]$ where $\kappa=\omega_2$ and GCH holds.

Definition. A ladder system $\langle C_\delta\mid\delta\in E\rangle$ is *reflected*, if there exist $\gamma\in S_1^2$ and a filtration $\langle X_i\mid i<\omega_1\rangle$ on γ such that $T=\{i<\omega_1\mid\sup(X_i)\in E \text{ and } C_{\sup(X_i)}\subseteq X_i\}$ is stationary. We also say that a ladder system is *non-reflecting*, if it is not reflected. Let $\langle C_\delta\mid\delta\in E\rangle$ be non-reflecting. Then we may associate a p.o.set Q which *shoots a club off* E. By this we mean that Q forces a club $\dot{C}$ in κ such that for any accumulation point α of $\dot{C}$, namely α is a limit ordinal and $\dot{C} \cap \alpha$ is cofinal in $\alpha \in \dot{C}$, we have $\alpha \notin E$. The conditions in Q are the possible initial segments of $\dot{C}$.

We argue in $V[G_\kappa]$. Let $\langle C_\delta \mid \delta \in E\rangle$ be a non-reflecting ladder system and Q be the associated p.o.set. Since there is no restrictions to put any new point above any condition in Q, it is clear that Q adds a cofinal and closed subset of κ. It is also clear that Q is of size $(2^{<\kappa})^{V[G_\kappa]} = (2^{\omega_1})^{V[G_\kappa]} = \kappa = \omega_2^{V[G_\kappa]}$. However it is not at all clear that Q is κ-Baire. Namely, Q does not add any new sequences of ordinals of length $< \kappa$. Before we start iterating, we present the following.

Observation. Let $\langle C_\delta \mid \delta \in E\rangle$ be non-reflecting in $V[G_\kappa]$ and let Q be the associated p.o.set in $V[G_\kappa]$ which shoots a club off E over $V[G_\kappa]$. Now we go back in V for a while. Let θ be a sufficiently large regular cardinal in V and N be an elementary substructure in V of $(H_\theta)^V$ such that $\kappa \in N$, $N \cap \kappa = \lambda$ is a strongly inaccessible cardinal in V, ${}^{<\lambda}N \subset N$ in V and $|N| = \lambda$ in V. We further assume that $\langle C_\delta \mid \delta \in E\rangle, Q \in N[G_\kappa]$ in $V[G_\kappa]$. Let M be the transitive collapse of N by the collapse π in V. Since $\mathrm{Lv}(\kappa,\omega_1)$ has the κ-c.c, every condition in $\mathrm{Lv}(\kappa,\omega_1)$ is $(\mathrm{Lv}(\kappa,\omega_1),N)$-generic. Hence π gets extended to π (same notation in use) collapsing $N[G_\kappa]$ onto $M[G_\lambda]$, where $G_\lambda = G_\kappa \cap \mathrm{Lv}(\lambda,\omega_1)$ is $\mathrm{Lv}(\lambda,\omega_1)$-generic over V. Notice that we may view $M[G_\lambda]$ as a generic extension of M via $\mathrm{Lv}(\lambda,\omega_1)$ over the transitive set model M. We also have that $V \cap {}^{<\lambda}M \subset M$ and $V[G_\lambda] \cap {}^{<\lambda}M[G_\lambda] \subset M[G_\lambda]$. Since $\langle C_\delta \mid \delta \in E\rangle \in N[G_\kappa]$, it gets collapsed to $\langle C_\delta \mid \delta \in E \cap \lambda\rangle \in M[G_\lambda]$. We claim that $E \cap \lambda$ is a non-stationary subset of $\lambda = \omega_2^{V[G_\lambda]}$ in $V[G_\lambda]$. This is because, if $E \cap \lambda$ were stationary in $V[G_\lambda]$. Then it is easy to see by genericity of $\dot{f}$ that for a (any) filtration $\langle X_i \mid i < \omega_1\rangle$ on λ in $V[G_{\lambda+1}] = V[G_\lambda][\dot{f}]$, where $\dot{f} : \omega_1 \longrightarrow \lambda$ onto, we have $T = \{i < \omega_1 \mid \sup(X_i) \in E \cap \lambda \text{ and } C_{\sup(X_i)} \subseteq X_i\}$ is stationary in $V[G_{\lambda+1}]$. This T remains stationary in $V[G_\kappa] = V[G_{\lambda+1}][G_{\lambda+1\kappa}]$, where $G_{\lambda+1\kappa}$ is $\mathrm{Lv}([\lambda+1,\kappa),\omega_1)$-generic over $V[G_{\lambda+1}]$. Hence the ladder system $\langle C_\delta \mid \delta \in E\rangle$ gets reflected. This would be a contradiction. Hence there is a club C of λ in $V[G_\lambda]$ such that $C \cap (E \cap \lambda) = \emptyset$. Now by making use of this C and the fact $V[G_\lambda] \cap {}^{<\lambda}M[G_\lambda] \subset M[G_\lambda]$, we may construct a $(\pi(Q), M[G_\lambda])$-generic sequence $\langle q_k \mid k < \lambda\rangle$ in $V[G_\lambda]$. Now take point-wise preimages of the q_k. Namely let $p_k \in Q \cap N[G_\kappa]$ such that $\pi(p_k) = q_k$. Then it is routine to show that $\langle p_k \mid k < \lambda\rangle$ is a $(Q, N[G_\kappa])$-generic sequence in $V[G_\kappa]$. Hence $\sup(\bigcup\{p_k \mid k < \lambda\}) = N[G_\kappa] \cap \kappa = N \cap \kappa = \lambda \notin E \subset S^2_0$. Hence $q = (\bigcup\{p_k \mid k < \lambda\}) \cup \{\lambda\} \in Q$ decides $\dot{O} \cap N[G_\kappa][\dot{O}] = \dot{O} \cap N[G_\kappa] = \{p \in Q \cap N[G_\kappa] \mid p \geq p_k$ for some $k < \lambda\} \in V[G_\kappa]$, where $\dot{O}$ are the Q-generic filters over $V[G_\kappa]$ with $q \in \dot{O}$. Hence Q is κ-Baire.

With this in mind, we are interested in the following class of preorders P in $V[G_\kappa]$.

Definition. A preorder P is *reasonable*, if P has a dense subset of size κ and is κ-Baire.

Proposition. Let P be reasonable in $V[G_\kappa]$. Then P preserves every cofinality, every cardinality and GCH.

Typically we will consider a $< \kappa$-support iterated forcing $P = P^*_{\alpha^*} \in (H_{\kappa^{++}})^{V[G_\kappa]}$ with $\alpha^* < (\kappa^+)^V = (\kappa^+)^{V[G_\kappa]} = \omega_3^{V[G_\kappa]}$. We intend to denote some of the objects in $V[G_\kappa]$ with * in this note.

Definition. A sequence $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^*\rangle$ (together with $\langle \dot{Q}^*_{\beta^*} \mid \beta^* < \alpha^*\rangle$, $\langle\langle \dot{C}^*_{\beta^*\delta} \mid \delta \in \dot{E}^*_{\beta^*}\rangle \mid \beta^* < \alpha^*\rangle$ and enumerations of names of the ladder systems from the intermediate stages $\langle(\alpha_1,\alpha_2) \mapsto \langle \dot{C}^{\alpha_1\alpha_2}_\delta \mid \delta \in \dot{E}^{\alpha_1\alpha_2}\rangle \mid \alpha_1 < \alpha^*, \alpha_2 < \kappa^+\rangle \in V[G_\kappa]$) is *our iteration*, if

- $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^*\rangle$ is a $< \kappa$-support iterated forcing with $\alpha^* < (\kappa^+)^V$ such that $P^*_{\beta^*+1} \equiv P^*_{\beta^*} * \dot{Q}^*_{\beta^*}$ for each $\beta^* < \alpha^*$.
- The support of $p^* \in P^*_{\beta^*}$ is defined by $\mathrm{supp}(p^*) = \{\xi < \beta^* \mid p^*(\xi) \neq \emptyset$ (as names)$\}$. And so $\mathrm{supp}(p^*)$ is of size $< \kappa$.
- For each $\beta^* < \alpha^*$, $P^*_{\beta^*}$ is reasonable and $\Vdash^{V[G_\kappa]}_{P^*_{\beta^*}}$ "$\langle \dot{C}^*_{\beta^*\delta} \mid \delta \in \dot{E}^*_{\beta^*}\rangle$ is a non-reflecting ladder system" and $\Vdash^{V[G_\kappa]}_{P^*_{\beta^*}}$ "the associated $\dot{Q}^*_{\beta^*}$ shoots a club off $\dot{E}^*_{\beta^*}$".

We would like to consider that the last preorder $P^*_{\alpha^*}$ has just finished its construction and waits to be explored its reasonability and more. Hence our iteration is known to be reasonable possibly except the last preorder.

We are interested in reasonable preorders and iterations in $(H_{\kappa^{++}})^{V[G_\kappa]}$.

Proposition. (Successor) Let $\mathcal{I} = \langle P^*_{\gamma^*} \mid \gamma^* \leq \beta^* + 1\rangle$ be our iteration. If $\langle P^*_{\gamma^*} \mid \gamma^* \leq \beta^*\rangle \in (H_{\kappa^{++}})^{V[G_\kappa]}$, then $P^*_{\beta^*+1} \in (H_{\kappa^{++}})^{V[G_\kappa]}$.

Proof. Since $P^*_{\beta^*}$ is reasonable, $P^*_{\beta^*}$ has a dense subset D of size κ and $P^*_{\beta^*}$ is κ-Baire. Since $1 \Vdash^{V[G_\kappa]}_{P^*_{\beta^*}}$ "$\dot{Q}^*_{\beta^*} \subset ([\kappa]^{<\kappa})^{V[G_\kappa]}$", we may represent each $p \in P^*_{\beta^*+1}$ as $p\lceil\beta^* \in P^*_{\beta^*}$ and $p(\beta^*) : [\kappa]^{<\kappa} \longrightarrow \mathcal{P}(D)$. Hence $|p(\beta^*)| \leq \kappa$ and $p(\beta^*) \subset [\kappa]^{<\kappa} \times \mathcal{P}(D) \subset (H_{\kappa^{++}})^{V[G_\kappa]}$. Hence $p(\beta^*) \in (H_{\kappa^{++}})^{V[G_\kappa]}$ and so $p \in (H_{\kappa^{++}})^{V[G_\kappa]}$. Hence $P^*_{\beta^*+1} \subset (H_{\kappa^{++}})^{V[G_\kappa]}$ and $|P^*_{\beta^*+1}| \leq |P^*_{\beta^*}| \times |{}^{[\kappa]^{<\kappa}}\mathcal{P}(D)| \leq \kappa^+$. Hence $P^*_{\beta^*+1} \in (H_{\kappa^{++}})^{V[G_\kappa]}$.

□

Proposition. (Limit) Let $\mathcal{I} = \langle P^*_{\gamma^*} \mid \gamma^* \leq \beta^*\rangle$ be our iteration with limit β^*. If for all $\gamma^* < \beta^*$, we have $P^*_{\gamma^*} \in (H_{\kappa^{++}})^{V[G_\kappa]}$, then $P^*_{\beta^*} \in (H_{\kappa^{++}})^{V[G_\kappa]}$.

Proof. For $p \in P^*_{\beta^*}$ and $\gamma^* < \beta^*$, we have $p\lceil\gamma^* \in (H_{\kappa^{++}})^{V[G_\kappa]}$. Hence $p \subset (H_{\kappa^{++}})^{V[G_\kappa]}$. But $|p| \leq \kappa$. Hence $p \in (H_{\kappa^{++}})^{V[G_\kappa]}$. Hence $P^*_{\beta^*} \subset (H_{\kappa^{++}})^{V[G_\kappa]}$.

Now if $\mathrm{cf}(\beta^*) < \kappa$, then $|P^*_{\beta^*}| \leq |(\kappa^+)^{<\kappa}| \leq \kappa^+$. Hence $P^*_{\beta^*} \in (H_{\kappa^{++}})^{V[G_\kappa]}$.

Next if $\mathrm{cf}(\beta^*) = \kappa$, then $|P^*_{\beta^*}| \leq \kappa \times \kappa^+ = \kappa^+$. Hence $P^*_{\beta^*} \in (H_{\kappa^{++}})^{V[G_\kappa]}$.

□

Corollary. For every our iteration $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^*\rangle$, we have $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^*\rangle \in (H_{\kappa^{++}})^{V[G_\kappa]}$.

Definition. For any reasonable $P \in (H_{\kappa^{++}})^{V[G_\kappa]}$, if $1 \Vdash^{V[G_\kappa]}_P$ "$\langle \dot{C}^P_\delta \mid \delta \in \dot{E}^P\rangle$ is a non-reflecting ladder system" for some $\langle \dot{C}^P_\delta \mid \delta \in \dot{E}^P\rangle \in (H_{\kappa^{++}})^{V[G_\kappa]}$, then we associate one of them to P. Let Φ denote this association. We see that $\Phi \subset (H_{\kappa^{++}})^{V[G_\kappa]} \in (H_{\kappa^{+++}})^{V[G_\kappa]}$. Hence $\Phi \in (H_{\kappa^{+++}})^{V[G_\kappa]} = (H_{\kappa^{+++}})^V[G_\kappa]$. Therefore we may fix a name $\dot{\Phi} \in (H_{\kappa^{+++}})^V$.

We think of $\dot{\Phi}$ as a name of a specific choice function. We may need to fix other names of choice functions $\cdots \in (H_{\kappa^{+++}})^V$ as we go along.

Definition. In V, let us fix $h : \kappa^+ \longrightarrow (\kappa^+) \times (\kappa^+)$ for book-keeping. Let $\mathcal{N}$ consists of N such that

- N is an elementary substructure of $(H_{\kappa^{+++}})^V$.
- $\kappa, h, \dot{\Phi}, \cdots \in N$.
- $N \cap \kappa = \lambda < \kappa$ and λ is a strongly inaccessible cardinal.
- ${}^{<\lambda}N \subset N$.
- $|N| = \lambda$.

Since κ is Mahlo, there are many elements in $\mathcal{N}$. We aim at the following.

Target. Let $N \in \mathcal{N}$ with $P^*_{\alpha^*} \in N[G_\kappa]$. Then for any $p \in P^*_{\alpha^*} \cap N[G_\kappa]$, there exists a $(P^*_{\alpha^*}, N[G_\kappa])$-generic sequence $\langle p^*_k \mid k < \lambda\rangle$ such that $\langle \pi(p^*_k) \mid k < \lambda\rangle \in V[G_\lambda]$, where π is the transitive collapse of $N[G_\kappa]$ onto $M[G_\lambda]$.

Definition. Our iteration $\mathcal{I} = \langle P^*_{\beta^*} \mid \beta^* \leq \alpha^*\rangle$ is *wonderful*, if for any $N \in \mathcal{N}$ with $\mathcal{I} \in N[G_\kappa]$ (by this we mean that the other associated sequences of objects with our iteration are also assumed to be in $N[G_\kappa]$ and we may simply denote this as $P^*_{\alpha^*} \in N[G_\kappa]$), any $p^* \in P^*_{\alpha^*} \cap N[G_\kappa]$, there exists a $(P^*_{\alpha^*}, N[G_\kappa])$-generic sequence $\langle p^*_k \mid k < \lambda\rangle$ below p^* such that $\langle \pi(p^*_k) \mid k < \lambda\rangle \in V[G_\lambda]$, where $\lambda = N \cap \kappa$ and π is the transitive collapse of $N[G_\kappa]$ onto $M[G_\lambda]$.

Proposition. If $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^*\rangle$ is wonderful, then the last preorder $P^*_{\alpha^*}$ is reasonable.

Proof. Fix any $p^* \in P^*_{\alpha^*}$. Since κ is Mahlo, we may pick $N \in \mathcal{N}$ such that $P^*_{\alpha^*} \in N[G_\kappa]$ and $p^* \in N[G_\kappa]$. Let $\lambda = N \cap \kappa$. By assumption, we may pick a $(P^*_{\alpha^*}, N[G_\kappa])$-generic sequence $\langle p^*_k \mid k < \lambda\rangle \in V[G_\kappa]$ below p^*.

Claim. There exist $q^* \in P^*_{\alpha^*}$ and $\langle s^*_{\beta^*} \mid \beta^* \in N[G_\kappa] \cap \alpha^* = N \cap \alpha^*\rangle$ such that

- For all $k < \lambda$, we have $q^* \leq p^*_k$.
- $\mathrm{supp}(q^*) = N \cap \alpha^*$.
- Each $s^*_{\beta^*}$ is a cofinal and closed subset of λ with $\sup(s^*_{\beta^*}) = \lambda$.
- If $\beta^* \in N \cap \alpha^*$, then $q^*\lceil\beta^* \Vdash^{V[G_\kappa]}_{P^*_{\beta^*}}$ "$q^*(\beta^*) = s^*_{\beta^*} \cup \{\lambda\}$".

Since q^* gets classified by the $\langle s^*_{\beta^*} \mid \beta^* \in N \cap \alpha^*\rangle$ and there are at most κ-many such sequences, we conclude that $P^*_{\alpha^*}$ is reasonable.

Proof. For each $\beta^* \in N \cap \alpha^*$, let $s^*_{\beta^*} = \bigcup\{s \mid \exists k < \lambda \exists l\ \lambda > l \geq k\ \ p^*_l\lceil\beta^* \Vdash^{V[G_\kappa]}_{P^*_{\beta^*}}$ "$p^*_k(\beta^*) = s$"$\}$.

We construct $q^*\lceil\beta^*$ by recursion on $\beta^* \leq \alpha^*$ in $V[G_\kappa]$. Suppose $\beta^* < \alpha^*$ and for all $k < \lambda$, we have $q^*\lceil\beta^* \leq p^*_k\lceil\beta^*$. We want to specify $q^*(\beta^*)$.

We first assume $\beta^* \notin N$. Then let $q^*(\beta^*) = \emptyset$. Since each $p^*_k \in N[G_\kappa]$, we have $\mathrm{supp}(p^*_k) \subset N[G_\kappa] \cap \alpha* = N \cap \alpha^*$. Hence for all $k < \lambda$, we have $p^*_k(\beta^*) = \emptyset$ and so $q^*\lceil(\beta^*+1) \leq p^*_k\lceil(\beta^*+1)$.

We next assume $\beta^* \in N$. By assumption $P^*_{\beta^*}$ is κ-Baire and $\langle p^*_k\lceil\beta^* \mid k < \lambda\rangle$ is an induced $(P^*_{\beta^*}, N[G_\kappa])$-generic sequence. Hence for any $k < \lambda$, there exists l such that $k \leq l < \lambda$ and $p^*_l\lceil\beta^*$ decides the value of $p^*_k(\beta^*)$ to be some s. Let O_{β^*} be any $P^*_{\beta^*}$-generic filter over $V[G_\kappa]$ with $q^*\lceil\beta^* \in O_{\beta^*}$. Since $q^*\lceil\beta^*$ is below every $p^*_k\lceil\beta^*$, we have in $V[G_\kappa][O_{\beta^*}]$ that $\langle p^*_k(\beta^*) \mid k < \lambda\rangle$ is a $(\dot{Q}^*_{\beta^*}, N[G_\kappa][O_{\beta^*}])$-generic sequence. Hence we conclude $s^*_{\beta^*}$ is a cofinal and closed subset of $N[G_\kappa][O_{\beta^*}] \cap \kappa = N[G_\kappa] \cap \kappa = N \cap \kappa = \lambda$. Since $\lambda \in (S^2_1)^{V[G_\kappa]}$, we have $q^*\lceil\beta^* \Vdash^{V[G_\kappa]}_{P^*_{\beta^*}}$ "$(\bigcup\{p^*_k(\beta^*) \mid k < \lambda\}) \cup \{\lambda\} = s^*_{\beta^*} \cup \{\lambda\} \in \dot{Q}^*_{\beta^*}$". Let $q^*\lceil\beta^* \Vdash^{V[G_\kappa]}_{P^*_{\beta^*}}$ "$q^*(\beta^*) = s^*_{\beta^*} \cup \{\lambda\}$". Then for all $k < \lambda$, we have $q^*\lceil(\beta^*+1) \leq p^*_k\lceil(\beta^*+1)$. Since $\langle p^*_k \mid k < \lambda\rangle$ is a $(P^*_{\alpha^*}, N[G_\kappa])$-generic sequence, we have that for any $\beta^* \in N[G_\kappa] \cap \alpha^* = N \cap \alpha^*$, there exists $k < \lambda$ such that $p^*_k\lceil\beta^* \Vdash^{V[G_\kappa]}_{P^*_{\beta^*}}$ "$p^*_k(\beta^*) \neq \emptyset$". Hence $\mathrm{supp}(q^*) = N[G_\kappa] \cap \alpha^* = N \cap \alpha^*$ holds.

□

Notice that we did not make use of $\langle \pi(p^*_k) \mid k < \lambda\rangle \in V[G_\lambda]$ in the above.

Notation. Let $N \in \mathcal{N}$. Let $\pi : N[G_\kappa] \longrightarrow M[G_\lambda]$ be the transitive collapse. The images of ordinals α^*, preorders P^* and P^*-names $\dot{Q}^*$ etc under π will be denoted as $\alpha = \pi(\alpha^*)$, $P = \pi(P^*)$ and $\dot{Q} = \pi(\dot{Q}^*)$.

We prove the following two lemmas later in this note. We assume these two for the rest of this section to finish our proof of theorem.

Lemma. (Successor) Let $\mathcal{I} = \langle P^*_{\beta^*} \mid \beta^* \leq \alpha^*+1\rangle$ be our iteration. If $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^*\rangle$ is wonderful, then so is $\mathcal{I}$.

Lemma. (Limit) Let $\mathcal{I} = \langle P^*_{\beta^*} \mid \beta^* \leq \alpha^*\rangle$ be our iteration with limit α^*. If for all $\gamma^* < \alpha^*$, $\langle P^*_{\beta^*} \mid \beta^* \leq \gamma^*\rangle$ are wonderful, then so is $\mathcal{I}$.

Corollary. Every our iteration $\mathcal{I}$ is wonderful.

Proof. Since $P^*_0 = \{\emptyset\}$, it is trivial that $\langle P^*_0\rangle$ is wonderful. Hence by recursion we may conclude $\mathcal{I}$ is wonderful.

□

Assuming that we have done with these two, we may finish our proof.

Proof of theorem. We argue in two cases.

Case 1. There exist our iteration $\mathcal{I} = \langle P^*_{\beta^*} \mid \beta^* \leq \alpha^* \rangle$ and $p \in P^*_{\alpha^*}$ such that $p \Vdash^{V[G_\kappa]}_{P^*_{\alpha^*}}$ "FRP(ω_2)" holds. Now think of doing trivial iteration to satisfy the statement of the theorem. Hence we are done.

Case 2. For any our iteration $\mathcal{I} = \langle P^*_{\beta^*} \mid \beta^* \leq \alpha^* \rangle$, we have $1 \Vdash^{V[G_\kappa]}_{P^*_{\alpha^*}}$ "FRP(ω_2) fails":

In this case, recall we have a fixed map $\Phi = \langle P \mapsto \langle \dot{C}^P_\delta \mid \delta \in \dot{E}^P \rangle \mid P \in (H_{\kappa^{++}})^{V[G_\kappa]}$ is a relevant reasonable preorder$\rangle$, where $1 \Vdash^{V[G_\kappa]}_P$ "the ladder system $\langle \dot{C}^P_\delta \mid \delta \in \dot{E}^P \rangle$ is non-reflecting".

Now we begin to construct a $< \kappa$-support iterated forcing $\langle P^*_{\alpha^*} \mid \alpha^* \leq (\kappa^+)^{V[G_\kappa]} \rangle$ by recursion on α^*. Suppose $\alpha^* < \kappa^+$ and that we have constructed $\mathcal{I} = \langle P^*_{\beta^*} \mid \beta^* \leq \alpha^* \rangle$ which is our iteration. Since the last preorder $P^*_{\alpha^*}$ is reasonable, we may fix an enumeration of names $\langle \langle \dot{C}^{\alpha^* \alpha_2}_\delta \mid \delta \in \dot{E}^{\alpha^* \alpha_2} \rangle \mid \alpha_2 < \kappa^+ \rangle$ of the ladder systems in $V[G_\kappa]^{P^*_{\alpha^*}}$ in addition to the fixed enumeration of suitable names of the ladder systems $\langle (\alpha_1, \alpha_2) \mapsto \langle \dot{C}^{\alpha_1 \alpha_2}_\delta \mid \delta \in \dot{E}^{\alpha_1 \alpha_2} \rangle \mid \alpha_1 < \alpha^*, \alpha_2 < \kappa^+ \rangle \in V[G_\kappa]$ in every intermediate stage $V[G_\kappa]^{P^*_{\alpha_1}}$ with $\alpha_1 < \alpha^*$.

It suffices to specify a non-reflecting ladder system $\langle \dot{C}^*_{\alpha^* \delta} \mid \delta \in \dot{E}^*_{\alpha^*} \rangle$ in $V[G_\kappa]^{P^*_{\alpha^*}} = V[G_\kappa][O_{\alpha^*}]$ as follows;

Let $h(\alpha^*) = (\alpha_1, \alpha_2)$. Hence $\alpha_1 \leq \alpha^*$ and $\alpha_2 < \kappa^+$. Take a look at $\langle \dot{C}^{\alpha_1 \alpha_2}_\delta \mid \delta \in \dot{E}^{\alpha_1 \alpha_2} \rangle$ in the current universe $V[G_\kappa][O_{\alpha^*}]$. If $\langle \dot{C}^{\alpha_1 \alpha_2}_\delta \mid \delta \in \dot{E}^{\alpha_1 \alpha_2} \rangle$ happens to be a non-reflecting ladder system in $V[G_\kappa][O_{\alpha^*}]$, then let $\langle \dot{C}^*_{\alpha^* \delta} \mid \delta \in \dot{E}^*_{\alpha^*} \rangle = \langle \dot{C}^{\alpha_1 \alpha_2}_\delta \mid \delta \in \dot{E}^{\alpha_1 \alpha_2} \rangle$. If $\langle \dot{C}^{\alpha_1 \alpha_2}_\delta \mid \delta \in \dot{E}^{\alpha_1 \alpha_2} \rangle$ does not happen to be non-reflecting in $V[G_\kappa][O_{\alpha^*}]$, then we switch to $\Phi(P^*_{\alpha^*})$ and let $\langle \dot{C}^*_{\alpha^* \delta} \mid \delta \in \dot{E}^*_{\alpha^*} \rangle = \Phi(P^*_{\alpha^*})$. In either case this specifies a non-reflecting ladder system $\langle \dot{C}_{\alpha^* \delta} \mid \delta \in \dot{E}^*_{\alpha^*} \rangle$. Now let us associate $\dot{Q}^*_{\alpha^*}$ which shoots a club off $\dot{E}^*_{\alpha^*}$.

Claim. $1 \Vdash^{V[G_\kappa]}_{P^*_{\kappa^+}}$ "FRP(ω_2)" holds.

Proof. Let O_{κ^+} be any $P^*_{\kappa^+}$-generic filter over $V[G_\kappa]$. Let us suppose on the contrary that $\langle C_\delta \mid \delta \in E \rangle$ were a non-reflecting ladder system in $V[G_\kappa][O_{\kappa^+}]$. Since $P^*_{\kappa^+}$ has the κ^+-c.c, we have $\alpha_1 < \kappa^+$ such that $\langle C_\delta \mid \delta \in E \rangle \in V[G_\kappa][O_{\alpha_1}]$, where $O_{\alpha_1} = O_{\kappa^+} \lceil \alpha_1$. Let $\alpha_2 < \kappa^+$ be such that $\langle C_\delta \mid \delta \in E \rangle$ is the interpretation of $\langle \dot{C}^{\alpha_1 \alpha_2}_\delta \mid \delta \in \dot{E}^{\alpha_1 \alpha_2} \rangle$ by O_{α_1}. Take $\alpha^* < \kappa^+$ such that $h(\alpha^*) = (\alpha_1, \alpha_2)$. Then $\langle C_\delta \mid \delta \in E \rangle$ is non-reflecting in the intermidiate $V[G_\kappa][O_{\alpha^*}]$. Hence $\dot{Q}^*_{\alpha^*}$ shoots a club off E. This contradicts to E being stationary in the final stage $V[G_\kappa][O_{\kappa^+}]$. Hence every ladder system must reflect in $V[G_\kappa][O_{\kappa^+}]$.

□

§3. Proof part one

Proof of lemma (Successor) We have our iteration $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^* + 1 \rangle$ such that $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^* \rangle$ is wonderful. We want to show that $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^* + 1 \rangle$ is wonderful.

Now let $N \in \mathcal{N}$ with $P^*_{\alpha^*+1} \in N[G_\kappa]$. Let $p^* \in P^*_{\alpha^*+1} \cap N[G_\kappa]$. We want a $(P^*_{\alpha^*+1}, N[G_\kappa])$-generic sequence $\langle p^*_k \mid k < \lambda \rangle \in V[G_\kappa]$ below p^* such that $\langle \pi(p^*_k) \mid k < \lambda \rangle \in V[G_\lambda]$. Since $P^*_{\alpha^*} \in N[G_\kappa]$ and $p^* \lceil \alpha^* \in P^*_{\alpha^*} \cap N[G_\kappa]$, we have a $(P^*_{\alpha^*}, N[G_\kappa])$-generic sequence $\langle q^*_k \mid k < \lambda \rangle \in V[G_\kappa]$ below $p^* \lceil \alpha^*$ such that $\langle \pi(q^*_k) \mid k < \lambda \rangle \in V[G_\lambda]$.

We denote $p = \pi(p^*)$, $q_k = \pi(q^*_k)$, $\alpha = \pi(\alpha^*)$, $\langle P_\beta \mid \beta \leq \alpha + 1 \rangle = \pi(\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^* + 1 \rangle)$ and $\overline{\langle q_k \mid k < \lambda \rangle} = \{ y \in P_\alpha \mid \exists k \ y \geq q_k \text{ in } P_\alpha \}$. Then in $V[G_\lambda]$, it is routine to show that $\langle q_k \mid k < \lambda \rangle$ is a $(P_\alpha, M[G_\lambda])$-generic sequence and so $\overline{\langle q_k \mid k < \lambda \rangle} \in V[G_\lambda]$ is a P_α-generic filter over $M[G_\lambda]$ with $p \lceil \alpha$ in it.

We have seen that there exists $q^* \in P^*_{\alpha^*}$ below the q^*_k's. Hence q^* is $(P^*_{\alpha^*}, N[G_\kappa])$-generic.

Let O_{α^*} be any $P^*_{\alpha^*}$-generic filter over $V[G_\kappa]$ with $q^* \in O_{\alpha^*}$. Let $Q^*_{\alpha^*}$ be the interpretation of $\dot{Q}^*_{\alpha^*}$ by O_{α^*}. Let $\langle C_\delta \mid \delta \in E \rangle$ be (omitting α^* and *) the interpretation of $\langle \dot{C}^*_{\alpha^* \delta} \mid \delta \in \dot{E}^*_{\alpha^*} \rangle$. Then $\langle C_\delta \mid \delta \in E \rangle$ is a non-reflecting ladder system and the associated $Q^*_{\alpha^*}$ shoots a club off E over $V[G_\kappa][O_{\alpha^*}]$. Note we have $Q^*_{\alpha^*} \in N[G_\kappa][O_{\alpha^*}]$ and $\langle C_\delta \mid \delta \in E \rangle \in N[G_\kappa][O_{\alpha^*}]$.

Then in the generic extension $V[G_\kappa][O_{\alpha^*}]$, the collapse $\pi : N[G_\kappa] \longrightarrow M[G_\lambda]$ gets extended to $\pi : N[G_\kappa][O_{\alpha^*}] \longrightarrow M[G_\lambda][\overline{\langle q_k \mid k < \lambda \rangle}]$. This is because $\{ \pi(x) \mid x \in O_{\alpha^*} \cap N[G_\kappa] \} = \{ \pi(x) \mid x \in N[G_\kappa], \exists k < \lambda \ x \geq q^*_k \text{ in } P^*_{\alpha^*} \} = \{ y \in P_\alpha \mid \exists k < \lambda \ y \geq q_k \text{ in } P_\alpha \}$.

We denote $\langle \dot{Q}_\beta \mid \beta \leq \alpha \rangle = \pi(\langle \dot{Q}^*_{\beta^*} \mid \beta^* \leq \alpha^* \rangle)$. Let Q_α be the interpretation of $\dot{Q}_\alpha$ by $\overline{\langle q_k \mid k < \lambda \rangle}$. Then we have $\langle C_\delta \mid \delta \in E \cap \lambda \rangle = \pi(\langle C_\delta \mid \delta \in E \rangle)$ and $Q_\alpha = \pi(Q^*_{\alpha^*})$. Hence $\langle C_\delta \mid \delta \in E \cap \lambda \rangle \in M[G_\lambda][\overline{\langle q_k \mid k < \lambda \rangle}]$ is a non-reflecting ladder system and Q_α is the associated p.o.set shooting a club off $E \cap \lambda$ over $M[G_\lambda][\overline{\langle q_k \mid k < \lambda \rangle}]$.

Claim. $E \cap \lambda$ is not stationary in $V[G_\lambda]$.

Proof. Suppose not. Then $\langle C_\delta \mid \delta \in E \cap \lambda \rangle$ is a ladder system in the intermidiate $V[G_\lambda]$. Hence it gets a filtration $\langle X_i \mid i < \omega_1 \rangle$ on λ in $V[G_{\lambda+1}]$. Then due to this filtration the original ladder system $\langle C_\delta \mid \delta \in E \rangle$ gets reflected in $V[G_\kappa][O_{\alpha^*}]$. This would be a contradiction.

□

Let $C \in V[G_\lambda]$ be a closed cofinal subset of λ such that $C \cap (E \cap \lambda) = C \cap E = \emptyset$. By making use of this C, we construct a $(P_{\alpha+1}, M[G_\lambda])$-generic sequence $\langle l \mapsto q_{k_l}^\frown \langle \tau_l \rangle \mid l < \lambda \rangle$ below $p \in P_{\alpha+1}$ in the intermidiate $V[G_\lambda]$. We first see that this suffices. Let $p^*_l \in P^*_{\alpha^*+1}$ be the preimage of $q_{k_l}^\frown \langle \tau_l \rangle \in P_{\alpha+1}$ under $\pi : N[G_\kappa] \longrightarrow M[G_\lambda]$. Then it is routine to show that this $\langle p^*_l \mid l < \lambda \rangle \in V[G_\kappa]$ is a $(P^*_{\alpha^*+1}, N[G_\kappa])$-generic sequence below p^*.

Now we begin to construct $q_{k_l}^\frown \langle \tau_l \rangle$ for $l < \lambda$ in $V[G_\lambda]$. Let $\langle D_l \mid l < \lambda \rangle$ enumerate the dense open subsets D of $P_{\alpha+1}$ with $D \in M[G_\lambda]$. The crutial fact is that $V[G_\lambda] \cap {}^{<\lambda}M[G_\lambda] \subset M[G_\lambda]$. This means that the initial segments constructed are all in $M[G_\lambda]$. Hence we may make use of the initial segments as sequences of conditions in $M[G_\lambda]$ and so may give rise to conditions in $P_{\alpha+1} \in M[G_\lambda]$.

($l = 0$): Since $q_0 \leq p \lceil \alpha$ in P_α, let $\tau_0 = p(\alpha)$. Then $q_0^\frown \langle \tau_0 \rangle \leq p$ in $P_{\alpha+1}$. Let $k_0 = 0$.

($l \longrightarrow l+1$): Suppose we have constructed $q_{k_l}^\frown \langle \tau_l \rangle \in P_{\alpha+1}$. Pick $q_{k'} \leq q_{k_l}$ so that $q_{k'}$ decides the value of τ_l to be s. This is possible as P_α is λ-Baire in $M[G_\lambda]$ and the q_k's form a $(P_\alpha, M[G_\lambda])$-generic sequence. Pick $e \in C$ with $\sup(s) < e < \lambda$. Then $q_{k'}^\frown \langle s \cup \{e\} \rangle \in P_{\alpha+1}$. Since $\{a \in P_\alpha \mid a \leq x \lceil \alpha$ for some $x \in D_l$ with $x \leq q_{k'}^\frown \langle s \cup \{e\} \rangle$ or (a is incompatible with $q_{k'}$ in P_α)$\}$ is dense open subset of P_α and belongs to $M[G_\lambda]$, we may pick $q_{k_{l+1}}^\frown \langle \tau_{l+1} \rangle \in D_l$ such that $q_{k_{l+1}}^\frown \langle \tau_{l+1} \rangle \leq q_{k'}^\frown \langle s \cup \{e\} \rangle \leq q_{l_k}^\frown \langle \tau_l \rangle$ in $P_{\alpha+1}$.

(Limit l): Suppose we have constructed $\langle q_{k_{l'}}^\frown \langle \tau_{l'} \rangle \mid l' < l \rangle$. Pick q_{k_l} so that for all $l' < l$, we have $q_{k_l} \leq q_{k_{l'}}$. Then q_{k_l} decides the value of $\sup(\bigcup\{\tau_{l'} \mid l' < l\})$ to be some limit $e' < \lambda$. Then $e' \in C$ and so $e' \notin E \cap \lambda$. Remember $E \cap \lambda$ is the relevant non-reflecting ladder system here in $M[G_\lambda][\overline{\langle q_k \mid k < \lambda \rangle}]$. Hence we may further assume $q_{k_l}^\frown \langle (\bigcup\{\tau_{l'} \mid l' < l\}) \cup \{e'\} \rangle \in P_{\alpha+1}$. Let $q_{k_l} \Vdash^{M[G_\lambda]}_{P_\alpha}$ "$\tau_l = (\bigcup\{\tau_{l'} \mid l' < l\}) \cup \{e'\}$". Then for all $l' < l$, we have $q_{k_l}^\frown \langle \tau_l \rangle \leq q_{k_{l'}}^\frown \langle \tau_{l'} \rangle$.

This completes the construction.

□

§4. Proof part two

Proof of lemma (Limit). Let $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^* \rangle$ be our iteration such that α^* is limit and for all $\gamma^* < \alpha^*$, we assume that $\langle P^*_{\beta^*} \mid \beta^* \leq \gamma^* \rangle$ are wonderful. We want to show that $\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^* \rangle$ is wonderful. We have seen that $P^*_{\alpha^*} \in (H_{\kappa^{++}})^{V[G_\kappa]}$. Let $N \in \mathcal{N}$ such that $P^*_{\alpha^*} \in N[G_\kappa]$. Let $p^* \in P^*_{\alpha^*} \cap N[G_\kappa]$.

We denote $p = \pi(p^*)$, $\alpha = \pi(\alpha^*)$, $\langle P_\beta \mid \beta \leq \alpha \rangle = \pi(\langle P^*_{\beta^*} \mid \beta^* \leq \alpha^* \rangle)$, $\langle \dot{Q}_\beta \mid \beta < \alpha \rangle = \pi(\langle \dot{Q}^*_{\beta^*} \mid \beta^* \leq \alpha^* \rangle)$, $\langle \langle \dot{C}_{\beta\delta} \mid \delta \in \dot{E}_\beta \rangle \mid \beta < \alpha \rangle = \pi(\langle \langle \dot{C}^*_{\beta^*\delta} \mid \delta \in \dot{E}^*_{\beta^*} \rangle \mid \beta^* < \alpha^* \rangle)$. We want a $(P_\alpha, M[G_\lambda])$-generic sequence in $V[G_\lambda]$.

For the rest of this section, we argue in the intermidiate $V[G_\lambda]$. Recall that $(\omega_1)^{V[G_\lambda]} = \omega_1^V$ and $(\omega_2)^{V[G_\lambda]} = \lambda$.

Claim. We have $\diamondsuit(S^2_1)$ in $V[G_\lambda]$.

Proof. Suppose that $A = (\dot{A})_{G_\lambda} \subseteq \lambda$ and $(\dot{C})_{G_\lambda}$ is a club in λ. In V, we may represent $\dot{A}$ as $\langle A_\alpha \mid \alpha < \lambda \rangle$ such that A_α is an anti-chain in $\mathrm{Lv}(\lambda, \omega_1)$ and so $|A_\alpha| < \lambda$. We assume $\alpha \in A$ iff $A_\alpha \cap G_\lambda \neq \emptyset$.

In V, let $C = \{\xi < \lambda \mid \forall\alpha < \xi\ A_\alpha \subset \mathrm{Lv}(\xi, \omega_1)\}$. Then this C is a club. Now in $V[G_\lambda]$, pick $\xi \in (\dot{C})_{G_\lambda} \cap C$ with $\mathrm{cf}(\xi) = \omega_1$. Then $A \cap \xi \in \mathcal{P}(\xi) \cap V[G_\xi]$ and $|\mathcal{P}(\xi) \cap V[G_\xi]| \leq \omega_1$. Hence $\langle \mathcal{P}(\xi) \cap V[G_\xi] \mid \xi \in S^2_1\rangle$ is a $\diamondsuit(S^2_1)$-sequence.

□

In view of $|M[G_\lambda]| = \lambda$ and $P_\alpha \cup \{P_\alpha\} \subset M[G_\lambda]$, we may fix $\langle i \mapsto (\langle q_{ij} \mid j < i\rangle, D(i)) \mid i \in S^2_1\rangle$ such that

- $\langle q_{ij} \mid j < i\rangle$ is a descending sequence of elements of P_α.
- $D(i) \subseteq P_{\xi_i}$ for some $\xi_i \leq \alpha$ and $D(i) \in M[G_\lambda]$.
- For any descending sequence $\langle p_i \mid i < \lambda\rangle$ of elements of P_α and any $D \subseteq P_\xi$ for some $\xi \leq \alpha$ with $D \in M[G_\lambda]$, the following

$$\{i \in S^2_1 \mid \langle q_{ij} \mid j < i\rangle = \langle p_j \mid j < i\rangle \text{ and } D(i) = D\}$$

is stationary.

We make use of this form of guessing to construct a $(P_\alpha, M[G_\lambda])$-generic sequence below p. We first take the greatest lower bound of $\langle q_{ji} \mid j < i\rangle$ as much as possible (i.e. q^0_i). Hence sort of $q^0_i \equiv \langle q_{ij}\lceil\alpha(i) \mid j < i\rangle$ and no more. Then we hit the possible dense open subset $D(i)$ below the lower bound in advance (i.e. q^1_i). Hence $q^1_i \leq q^0_i$ in $P_{\alpha(i)}$ and if $D(i)$ is dense open in P_{ξ_i} with some $\xi_i \leq \alpha(i)$, then $q^1_i\lceil\xi_i \in D(i)$. Therefore as long as guessing succeed, we would have taken care of every relevant dense open subset. This way we cover shortages of steps compared to the number of relevent dense open subsets (i.e. ω, ω_1 vs. ω_2).

Definition. We associate $\langle i \mapsto (q^0_i, q^1_i, \alpha(i)) \mid i \in S^2_1\rangle$ such that

- $\alpha(i) \leq \alpha$ and $q^0_i, q^1_i \in P_{\alpha(i)}$.
- For any $j < i$ and any $\eta < \alpha(i)$, we have $q^0_i\lceil\eta \leq q_{ij}\lceil\eta$ in P_η and $q^0_i\lceil\eta$ forces (over $M[G_\lambda]$) the following;

$$q^0_i(\eta) = \overline{\bigcup\{q_{ij}(\eta) \mid j < i\}},$$

where $\overline{s}$ denotes the closure of s. Therefore, $q^0_i\lceil\eta$ forces the disjunction of the following (1) or (2);

(1) $\exists j < i\ q_{ji}(\eta) \neq \emptyset$ and $\sup(\bigcup\{q_{ij}(\eta) \mid j < i\}) \notin \dot{E}_\eta$ and

$$q^0_i(\eta) = (\bigcup\{q_{ij}(\eta) \mid j < i\}) \cup \{\sup(\bigcup\{q_{ij}(\eta) \mid j < i\})\}.$$

(2) $\forall j < i\ q_{ij}(\eta) = \emptyset$ and

$$q^0_i(\eta) = \emptyset.$$

- If $\alpha(i) < \alpha$, then q^0_i fails to force the disjunction of (1) or (2) as above.
- If $D(i)$ is a dense open subset of P_{ξ_i} with some $\xi_i \leq \alpha(i)$, then $q^1_i\lceil\xi_i \in D(i)$ and $q^1_i \leq q^0_i$ in $P_{\alpha(i)}$. Otherwise, $q^1_i = q^0_i$.

  Note that we have $\langle q_{ij} \mid j < i\rangle \in M[G_\lambda]$ and $\mathrm{supp}(q^0_i) \subseteq \bigcup\{\mathrm{supp}(q_{ij}) \mid j < i\}$ and so of size $< \lambda$.

Definition. Let $\phi(\xi, \langle p_i \mid i < \lambda\rangle, C, a)$ stands for the following;

- $\xi \leq \alpha$.
- $\langle p_i \mid i < \lambda\rangle$ is a descending sequence of elements in P_ξ below $a \in P_\xi$ and C is a club in λ.
- For any $i \in C \cap S^2_1$ and any $\eta < \xi$, $p_i\lceil\eta$ forces (over $M[G_\lambda]$) the following;

$$p_i(\eta) = \overline{\bigcup\{p_j(\eta) \mid j < i\}}.$$

- For any $i \in C \cap S^2_1$, if $\langle p_j \mid j < i\rangle \equiv \langle q_{ij}\lceil\xi \mid j < i\rangle$, then we have

$$p_{i+1} \leq q^1_i\lceil\xi \text{ in } P_\xi,$$

where $\langle p_j \mid j < i\rangle \equiv \langle q_{ij}\lceil\xi \mid j < i\rangle$ means $\forall j < i \exists j' < i\ q_{ij'}\lceil\xi \leq p_j$ in P_ξ and conversely $\forall j < i \exists j' < i\ p_{j'} \leq q_{ij}\lceil\xi$ in P_ξ. Hence these two sequences are not required to be literally equal but share the same strength.

We may abbreviate the third condition in the above as $p_i \equiv \langle p_j \mid j < i\rangle$.

Proposition. If $\phi(\xi, \langle p_i \mid i < \lambda\rangle, C, w)$, $i \in C \cap S^2_1$ and $\langle p_j \mid j < i\rangle \equiv \langle q_{ij}\lceil\xi \mid j < i\rangle$, then $\xi \leq \alpha(i)$ and $p_i \equiv q^0_i\lceil\xi$.

Proof. It is routine to show $p_i\lceil\eta \equiv q^0_i\lceil\eta$ by induction on $\eta \leq \xi$.

□

Note that we did not make use of the 4th condition of $\phi(\xi, \langle p_i \mid i < \lambda\rangle, C, w)$ in the proof. And by this proposition, the 4th condition makes sense.

Proposition. If $\phi(\xi, \langle p_i \mid i < \lambda\rangle, C, w)$, then $\langle p_i \mid i < \lambda\rangle$ is a $(P_\xi, M[G_\lambda])$-generic sequence below w.

Proof. Let D be any dense open subset of P_ξ with $D \in M[G_\lambda]$. By assumption on $\langle(\langle q_{ij} \mid j < i\rangle, D(i)) \mid i \in S^2_1\rangle$, we may pick $i \in C \cap S^2_1$ such that $D = D(i)$ and $\langle p_j^\frown 1 \mid j < i\rangle = \langle q_{ij} \mid j < i\rangle$. Hence $\langle p_j \mid j < i\rangle = \langle q_{ij}\lceil\xi \mid j < i\rangle$ and $D(i)$ is dense open in P_ξ. Hence $p_{i+1} \leq q^1_i\lceil\xi$ and $q^1_i\lceil\xi \in D(i)$. Hence $p_{i+1} \in D(i) = D$.

□

Definition. Let $\phi(\eta, \langle p^\eta_i \mid i < \lambda\rangle, C^\eta, a)$ and $\phi(\xi, \langle p^\xi_i \mid i < \lambda\rangle, C^\xi, b)$. We write

$$(\eta, \langle p^\eta_i \mid i < \lambda\rangle, C^\eta, a)\, R\, (\xi, \langle p^\xi_i \mid i < \lambda\rangle, C^\xi, b),$$

if

- $\eta < \xi$, $C^\eta \supseteq C^\xi$ and $a = b\lceil\eta$.
- $\forall i < \lambda \exists j \geq i\ p^\xi_i\lceil\eta = p^\eta_j$.
- There exists a club $C_{\eta\xi}$ in λ such that
  (1) $C_{\eta\xi} \subseteq C^\eta \cap C^\xi$.
  (2) $\forall i \in C_{\eta\xi} \cap S^2_1\ p^\eta_i = p^\xi_i\lceil\eta$.

Proposition. R is transitive.

Proof. $(\eta_1, \langle p^1_i \mid i < \lambda\rangle, C_1, a_1)\, R\, (\eta_2, \langle p^2_i \mid i < \lambda\rangle, C_2, a_2)\, R\, (\eta_3, \langle p^3_i \mid i < \lambda\rangle, C_3, a_3)$ implies $(\eta_1, \langle p^1_i \mid i < \lambda\rangle, C_1, a_1)\, R\, (\eta_3, \langle p^3_i \mid i < \lambda\rangle, C_3, a_3)$.

□

Work in $V[G_\lambda]$. By induction on $\xi \leq \alpha = \pi(\alpha^*)$, we show the following IH(ξ);

$\forall\eta < \xi\, \forall\langle p^\eta_i \mid i < \lambda\rangle\, \forall C^\eta\, \forall w \in P_\xi$, if $\phi(\eta, \langle p^\eta_i \mid i < \lambda\rangle, C^\eta, w\lceil\eta)$, then there exists $(\langle p^\xi_i \mid i < \lambda\rangle, C^\xi)$ such that

- $\phi(\xi, \langle p^\xi_i \mid i < \lambda\rangle, C^\xi, w)$.
- $(\eta, \langle p^\eta_i \mid i < \lambda\rangle, C^\eta, w\lceil\eta)\, R\, (\xi, \langle p^\xi_i \mid i < \lambda\rangle, C^\xi, w)$.

In particular, let $\eta = 0$, $\xi = \alpha$ and $w = p = \pi(p^*) \in P_\alpha$. Since $\phi(0, \langle\emptyset \mid i < \lambda\rangle, \lambda, w\lceil 0)$ holds, we have $(\langle p^\alpha_i \mid i < \lambda\rangle, C^\alpha)$ such that $\phi(\alpha, \langle p^\alpha_i \mid i < \lambda\rangle, C^\alpha, p)$. Hence $\langle p^\alpha_i \mid i < \lambda\rangle \in V[G_\lambda]$ is a $(P_\alpha, M[G_\lambda])$-generic sequence below p. This completes the proof of lemma (Limit).

§5. Proof part three

Proof of IH(ξ) by induction.

IH(0): IH(0) is vacuously true.

We have two remaining cases.

IH(ξ) implies IH($\xi+1$): Since R is transitive, we may assume that $\eta = \xi$. Suppose $\phi(\xi, \langle p_i^\xi \mid i < \lambda\rangle, C^\xi, w\lceil\xi)$ and $w \in P_{\xi+1}$. We want $\langle p_i^{\xi+1} \mid i < \lambda\rangle$ and $C^{\xi+1}$ such that $\phi(\xi+1, \langle p_i^{\xi+1} \mid i < \lambda\rangle, C^{\xi+1}, w)$ and $(\xi, \langle p_i^\xi \mid i < \lambda\rangle, C^\xi, w\lceil\xi)\, R\, (\xi+1, \langle p_i^{\xi+1} \mid i < \lambda\rangle, C^{\xi+1}, w)$.

Remember that we have the transitive collapse $\pi : N[G_\kappa] \longrightarrow M[G_\lambda]$. Let $\pi(\xi^*) = \xi$ and $\pi(p_i') = p_i^\xi$ for each $i < \lambda$. Hence $\pi(P^*_{\xi^*}) = P_\xi$. Since $\langle p_i^\xi \mid i < \lambda\rangle$ is a $(P_\xi, M[G_\lambda])$-generic sequence, its pointwise preimages $\langle p_i' \mid i < \lambda\rangle \in V[G_\kappa]$ is a $(P^*_{\xi^*}, N[G_\kappa])$-generic sequence with $\mathrm{cf}(\lambda) = \omega_1$ in $V[G_\kappa]$. We know that there exists a lower bound $q' \in P^*_{\xi^*}$ of the p_i''s. This q' is $(P^*_{\xi^*}, N[G_\kappa])$-generic.

Let O_{ξ^*} be $P^*_{\xi^*}$-generic over $V[G_\kappa]$ with $q' \in O_{\xi^*}$. Then in the generic extension $V[G_\kappa][O_{\xi^*}]$, we have the extension $\pi : N[G_\kappa][O_{\xi^*}] \longrightarrow M[G_\lambda][\langle p_i^\xi \mid i < \lambda\rangle]$.

Let $\langle C_\delta \mid \delta \in E\rangle$ be the interpretation of $\langle \dot{C}^*_{\xi^*\delta} \mid \delta \in \dot{E}^*_{\xi^*}\rangle$ by O_{ξ^*}. Then $\langle C_\delta \mid \delta \in E\rangle \in N[G_\kappa][O_{\xi^*}]$ and $\pi(\langle C_\delta \mid \delta \in E\rangle) = \langle C_\delta \mid \delta \in E \cap \lambda\rangle$ holds. Since $\langle C_\delta \mid \delta \in E\rangle$ is non-reflecting in $V[G_\kappa][O_{\xi^*}]$, it must hold that $E \cap \lambda$ is not stationary in $V[G_\lambda]$. This is because if $E \cap \lambda$ were stationary in $V[G_\lambda]$. Then $\langle C_\delta \mid \delta \in E \cap \lambda\rangle$ gets a filtration on λ which reflets $\langle C_\delta \mid \delta \in E \cap \lambda\rangle$ in $V[G_{\lambda+1}]$. This filtration remains up $V[G_\kappa]$ and further up $V[G_\kappa][O_{\xi^*}]$. This contradicts that $\langle C_\delta \mid \delta \in E\rangle$ is non-reflecting in this last $V[G_\kappa][O_{\xi^*}]$.

Since $E \cap \lambda$ is not stationary in $V[G_\lambda]$, we may pick a club $C \in V[G_\lambda]$ such that $C \cap (E \cap \lambda) = \emptyset$. Let $\dot{E}_\xi = \pi(\dot{E}^*_{\xi^*})$. Then this $\dot{E}_\xi$ is a P_ξ-name in $M[G_\lambda]$ such that $E \cap \lambda$ is the interpretation of $\dot{E}_\xi$ by $\langle p_i^\xi \mid i < \lambda\rangle$.

We work in $V[G_\lambda]$. The crutial point was $V[G_\lambda] \cap {}^{<\lambda}M[G_\lambda] \subset M[G_\lambda]$ and $P_{\xi+1} \in M[G_\lambda]$. We construct $\langle p_{i_k}^\xi{}^\frown\langle\tau_k\rangle \mid k < \lambda\rangle$ by recursion on $k < \lambda$.

Case ($k=0$): Let $p_{i_0}^\xi{}^\frown\langle\tau_0\rangle \le w$ in $P_{\xi+1}$.

Case (k to $k+1$): Suppose we have constructed $p_{i_k}^\xi{}^\frown\langle\tau_k\rangle \in P_{\xi+1}$. Want $p_{i_{k+1}}^\xi{}^\frown\langle\tau_{k+1}\rangle \in P_{\xi+1}$.

Subcase 1. k is either 0 or successor: Pick a large $i_{k+1} < \lambda$ and τ_{k+1} such that $p_{i_{k+1}}^\xi \Vdash_{P_\xi}^{M[G_\lambda]}$ "$\max(\tau_k) < e < \max(\tau_{k+1})$" for some $e \in C$.

Subcase 2. k is limit: We have two cases.

Subsubcase 2.1. $i_k = k \in C^\xi \cap S_1^2$ and $\langle p_{i_{k'}}^\xi{}^\frown\langle\tau_{k'}\rangle \mid k' < k\rangle \equiv \langle q_{kk'}\lceil(\xi+1) \mid k' < k\rangle$: Then we have $\xi + 1 \le \alpha(k)$ and $p_k^\xi \equiv q_k^0\lceil\xi$. By subcase 2 below, we have $p_k^\xi \Vdash_{P_\xi}^{M[G_\lambda]}$ "$\tau_k = (\bigcup\{\tau_{k'} \mid k' < k\}) \cup \{\sup(\bigcup\{\tau_{k'} \mid k' < k\})\} = q_k^0(\xi)$", $q_k^1 \le q_k^0$ in $P_{\alpha(k)}$ and $p_{k+1}^\xi \le q_k^1\lceil\xi$ holds. Let us take $\tau_{k+1} = q_k^1(\xi)$. Then $p_{k+1}^\xi{}^\frown\langle\tau_{k+1}\rangle \le q_k^1\lceil(\xi+1), p_k^\xi{}^\frown\langle\tau_k\rangle$. Let $i_{k+1} = k+1$. Hence $p_{i_{k+1}}^\xi{}^\frown\langle\tau_{k+1}\rangle = p_{k+1}^\xi{}^\frown\langle\tau_{k+1}\rangle$.

Subsubcase 2.2. Otherwise: Take $p_{i_{k+1}}^\xi{}^\frown\langle\tau_{k+1}\rangle \le p_{i_k}^\xi{}^\frown\langle\tau_k\rangle$ as in Subcase 1.

Case (k is limit): We have constructed $p_{i_{k'}}^\xi{}^\frown\langle\tau_{k'}\rangle$ for all $k' < k$. We want $p_{i_k}^\xi{}^\frown\langle\tau_k\rangle$.

Subcase 1. $\mathrm{cf}(k) = \omega$: Pick $i_k < \lambda$ so that for all $k' < k$, $i_{k'} < i_k$. Then for all $k' < k$, we have $p_{i_k}^\xi \le p_{i_{k'}}^\xi$. Since $E \cap \lambda = \{\nu < \lambda \mid \exists l < \lambda\ p_l^\xi \Vdash_{P_\xi}^{M[G_\lambda]}$ "$\nu \in \dot{E}_\xi$"$\}$, we may assume that $p_{i_k}^\xi \Vdash_{P_\xi}^{M[G_\lambda]}$ "$\sup(\bigcup\{\tau_{k'} \mid k' < k\}) \notin \dot{E}_\xi$", where $\dot{E}_\xi = \pi(\dot{E}^*_{\xi^*})$. Hence we may pick τ_k so that $p_{i_k}^\xi \Vdash_{P_\xi}^{M[G_\lambda]}$ "$\tau_k = (\bigcup\{\tau_{k'} \mid k' < k\}) \cup \{\sup(\bigcup\{\tau_{k'} \mid k' < k\})\} \in \dot{Q}_\xi$", where $\dot{Q}_\xi = \pi(\dot{Q}^*_{\xi^*})$. For all $k' < k$, we have $p_{i_k}^\xi{}^\frown\langle\tau_k\rangle \le p_{i_{k'}}^\xi{}^\frown\langle\tau_{k'}\rangle$.

Subcase 2. $\mathrm{cf}(k) = \omega_1$: Let $i_k = \sup\{i_{k'} \mid k' < k\}$. Then for all $k' < k$, we have $p^\xi_{i_k} \le p^\xi_{i_{k'}}$ and $p_{i_k} \Vdash^{M[G_\lambda]}_{P_\xi}$ "$\sup(\bigcup\{\tau_{k'} \mid k' < k\}) \in S^2_1$". Hence may take τ_k to be such that $p^\xi_{i_k} \Vdash^{M[G_\lambda]}_{P_\xi}$ "$\tau_k = (\bigcup\{\tau_{k'} \mid k' < k\}) \cup \{\sup(\bigcup\{\tau_{k'} \mid k' < k\})\} \in \dot{Q}_\xi$".

This completes the construction of $\langle p^\xi_{i_k} {}^\frown \langle \tau_k \rangle \mid k < \lambda \rangle$.
Let $C^{\xi+1} = C^\xi \cap \{k < \lambda \mid \forall k' < k \; i_{k'} < k\}$. Then this $C^{\xi+1} \in V[G_\lambda]$ is a club in λ.

Claim. If $k \in C^{\xi+1} \cap S^2_1$, then $i_k = k$ holds.

Proof. Since $\langle i_k \mid k < \lambda \rangle$ is strictly increasing, we have $k \le i_k$. Since $i_{k'} < k$ for all $k' < k$ and $\mathrm{cf}(k) = \omega_1$, we have $i_k = \sup\{i_{k'} \mid k' < k\} \le k$. Hence $i_k = k$.

□

Now for each $k < \lambda$, let us set

$$p^{\xi+1}_k = p^\xi_{i_k} {}^\frown \langle \tau_k \rangle.$$

We want to show $\phi(\xi + 1, \langle p^{\xi+1}_k \mid k < \lambda \rangle, C^{\xi+1}, w)$ and $(\xi, \langle p^\xi_k \mid k < \lambda \rangle, C^\xi, w\lceil\xi) \, R \, (\xi + 1, \langle p^{\xi+1}_k \mid k < \lambda \rangle, C^{\xi+1}, w)$.

By construction we have that $\langle p^{\xi+1}_k \mid k < \lambda \rangle$ is descending below w in $P_{\xi+1}$ and that $C^{\xi+1}$ is a club in λ.
Let $k \in C^{\xi+1} \cap S^2_1$. Then we have $k = i_k$. It is routine to check that for any $\eta < \xi + 1$, $p^{\xi+1}_k \lceil \eta$ forces the following;

$$p^{\xi+1}_k(\eta) = \overline{\bigcup\{p^{\xi+1}_{k'}(\eta) \mid k' < k\}}.$$

(details) Let $\eta < \xi$. Then $p^{\xi+1}_k \lceil \eta = p^\xi_k \lceil \eta$ which forces the disjunction of (1) or (2);

(1) $\exists k' < k \; p^{\xi+1}_{k'}(\eta) = p^\xi_{i_{k'}}(\eta) \ne \emptyset$, $\sup(\bigcup\{p^{\xi+1}_{k'}(\eta) \mid k' < k\}) = \sup(\bigcup\{p^\xi_{k'}(\eta) \mid k' < k\}) \notin \dot{E}_\eta$ and $p^{\xi+1}_k(\eta) = p^\xi_k(\eta) = (\bigcup\{p^\xi_{k'}(\eta) \mid k' < k\}) \cup \{\sup(\bigcup\{p^\xi_{k'}(\eta) \mid k' < k\})\} = (\bigcup\{p^\xi_{i_{k'}}(\eta) \mid k' < k\}) \cup \{\sup(\bigcup\{p^\xi_{i_{k'}}(\eta) \mid k' < k\})\} = (\bigcup\{p^{\xi+1}_{k'}(\eta) \mid k' < k\}) \cup \{\sup(\bigcup\{p^{\xi+1}_{k'}(\eta) \mid k' < k\})\}$.
(2) $\forall k' < k \; p^{\xi+1}_{k'}(\eta) = p^\xi_{i_{k'}}(\eta) = \emptyset$ and $p^{\xi+1}_k(\eta) = p^\xi_k(\eta) = \emptyset$.

Next let $\eta = \xi$. Then $p^{\xi+1}_k \lceil \xi = p^\xi_k$ which forces the following (1);

(1) $\exists k' < k \; p^{\xi+1}_{k'}(\xi) = \tau_{k'} \ne \emptyset$, $\sup(\bigcup\{p^{\xi+1}_{k'}(\xi) \mid k' < k\}) = \sup(\bigcup\{\tau_{k'} \mid k' < k\}) \notin \dot{E}_\eta$ and $p^{\xi+1}_k(\xi) = \tau_k = (\bigcup\{\tau_{k'} \mid k' < k\}) \cup \{\sup(\bigcup\{\tau_{k'} \mid k' < k\})\} = (\bigcup\{p^{\xi+1}_{k'}(\xi) \mid k' < k\}) \cup \{\sup(\bigcup\{p^{\xi+1}_{k'}(\xi) \mid k' < k\})\}$.

Next suppose $k \in C^{\xi+1} \cap S^2_1$ and that $\langle p^{\xi+1}_{k'} \mid k' < k \rangle \equiv \langle q_{kk'} \lceil (\xi + 1) \mid k' < k \rangle$. Then $i_k = k \in C^\xi \cap S^2_1$ and $\langle p^\xi_{i_{k'}} {}^\frown \langle \tau_{k'} \rangle \mid k' < k \rangle \equiv \langle q_{kk'} \lceil (\xi + 1) \mid k' < k \rangle$. Hence $p^{\xi+1}_{k+1} = p^\xi_{k+1} {}^\frown \langle \tau_{k+1} \rangle \le q^1_k \lceil (\xi + 1)$, as $k + 1 = i_{k+1}$. Therefore we have $\phi(\xi + 1, \langle p^{\xi+1}_k \mid k < \lambda \rangle, C^{\xi+1}, w)$.

Lastly, $(\xi, \langle p^\xi_i \mid i < \lambda \rangle, C^\xi, w\lceil\xi) \, R \, (\xi + 1, \langle p^{\xi+1}_k \mid k < \lambda \rangle, C^{\xi+1}, w)$ holds.

(details) $\xi < \xi + 1$, $C^\xi \supseteq C^{\xi+1}$, $w\lceil\xi = w\lceil\xi$.

$\forall k < \lambda \exists i_k \ge k \; p^\xi_{i_k} = p^{\xi+1}_k \lceil \xi$.

Let $C_{\xi\xi+1} = C^{\xi+1}$. Then for $k \in C_{\xi\xi+1} \cap S^2_1$, we have $p^\xi_k = p^\xi_{i_k} = p^{\xi+1}_k \lceil \xi$, as $k \in C^{\xi+1} \cap S^2_1$ implies $i_k = k$.

This completes $\mathrm{IH}(\xi)$ implies $\mathrm{IH}(\xi + 1)$.

§6. Proof part four

γ limit, $(\forall \xi < \gamma \; \mathrm{IH}(\xi))$ implies $\mathrm{IH}(\gamma)$: We still work in $V[G_\lambda]$. Let $\gamma \le \alpha$ and γ be limit. We show that $(\forall \xi < \gamma \; \mathrm{IH}(\xi))$ implies $\mathrm{IH}(\gamma)$. We have two cases according to $\mathrm{cf}(\gamma) = \omega, \omega_1$ and to $\mathrm{cf}(\gamma) = \omega_2$.

Case. $\mathrm{cf}(\gamma) = \omega, \omega_1$: Let $\eta < \gamma$, $w \in P_\gamma$ and $\phi(\eta, \langle p_i^\eta \mid i < \lambda\rangle, C^\eta, w\lceil\eta)$. We want $\langle p_i^\gamma \mid i < \lambda\rangle$ and C^γ such that $\phi(\gamma, \langle p_i^\gamma \mid i < \lambda\rangle, C^\gamma, w)$ and $(\eta, \langle p_i^\eta \mid i < \lambda\rangle, C^\eta, w\lceil\eta)\, R\, (\gamma, \langle p_i^\gamma \mid i < \lambda\rangle, C^\gamma, w)$.

To this end let $\langle \gamma_k \mid k \le \mathrm{cf}(\gamma)\rangle$ be a strictly $<$-increasing continuous sequence of ordinals such that $\gamma_0 = \eta$ and $\gamma_{\mathrm{cf}(\gamma)} = \gamma$. It suffices construct $\langle p_i^{\gamma_k} \mid i < \lambda\rangle$ and C^{γ_k} by recursion on $k \le \mathrm{cf}(\gamma)$ such that $\phi(\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda\rangle, C^{\gamma_k}, w\lceil\gamma_k)$ and for all $l < k$, we have $(\gamma_l, \langle p_i^{\gamma_l} \mid i < \lambda\rangle, C^{\gamma_l}, w\lceil\gamma_l)\, R\, (\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda\rangle, C^{\gamma_k}, w\lceil\gamma_k)$.

<u>$k = 0$</u>: Let $\langle p_i^{\gamma_0} \mid i < \lambda\rangle = \langle p_i^\eta \mid i < \lambda\rangle$ and $C^{\gamma_0} = C^\eta$.

<u>k to $k+1$</u>: Suppose we have constructed $\langle p_i^{\gamma_k} \mid i < \lambda\rangle$ and C^{γ_k} such that $\phi(\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda\rangle, C^{\gamma_k}, w\lceil\gamma_k)$. By $\mathrm{IH}(\gamma_{k+1})$, we have $\langle p_i^{\gamma_{k+1}} \mid i < \lambda\rangle$ and $C^{\gamma_{k+1}}$ such that $\phi(\gamma_{k+1}, \langle p_i^{\gamma_{k+1}} \mid i < \lambda\rangle, C^{\gamma_{k+1}}, w\lceil\gamma_{k+1})$ and $(\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda\rangle, C^{\gamma_k}, w\lceil\gamma_k)\, R\, (\gamma_{k+1}, \langle p_i^{\gamma_{k+1}} \mid i < \lambda\rangle, C^{\gamma_{k+1}}, w\lceil\gamma_{k+1})$.

<u>k limit</u>: Let

$$C^{\gamma_k 0} = \bigcap\{C_{\gamma_l\gamma_m} \mid l < m < k\}$$

and for each $i \in S_1^2 \cap C^{\gamma_k 0}$, let

$$p_i^{\gamma_k 0} = \bigcup\{p_i^{\gamma_l} \mid l < k\}.$$

Then $p_i^{\gamma_k 0} \in P_{\gamma_k}$, as $V[G_\lambda] \cap {}^{<\lambda}M[G_\kappa] \subset M[G_\lambda]$ and $|\operatorname{supp}(p_i^{\gamma_k 0})| \le \omega_1$.

Let $f : \lambda \longrightarrow S_1^2 \cap C^{\gamma_k 0}$ be the $\in$-isomorphism and let $C(f) = \{i < \lambda \mid \forall j < i\ f(j) < i\}$. Let

$$C^{\gamma_k} = C^{\gamma_k 0} \cap C(f)$$

and for each $i < \lambda$, let

$$p_i^{\gamma_k} = p_{f(i)}^{\gamma_k 0}.$$

Note that if $i \in S_1^2 \cap C^{\gamma_k}$, then $f(i) = i$ holds.

Claim. We have that $\phi(\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda\rangle, C^{\gamma_k}, w\lceil\gamma_k)$ and for all $l < k$, we have

$$(\gamma_l, \langle p_i^{\gamma_l} \mid i < \lambda\rangle, C^{\gamma_l}, w\lceil\gamma_l)\, R\, (\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda\rangle, C^{\gamma_k}, w\lceil\gamma_k).$$

Proof. Some details. ($p_i^{\gamma_k}$ are descending): Let $i_1 < i_2$. Then $p_{i_1}^{\gamma_k} = p_{f(i_1)}^{\gamma_k 0} \ge p_{f(i_2)}^{\gamma_k 0} = p_{i_2}^{\gamma_k}$.

(For $i \in S_1^2 \cap C^{\gamma_k}$, $p_i^{\gamma_k} \equiv \langle p_j^{\gamma_k} \mid j < i\rangle$): Let $i \in S_1^2 \cap C^{\gamma_k}$. Let $\rho < \gamma_k$. Want that $p_i^{\gamma_k}\lceil\rho$ forces the following;

$$p_i^{\gamma_k}(\rho) = \overline{\bigcup\{p_j^{\gamma_k}(\rho) \mid j < i\}}.$$

To see this, pick $l < k$ such that $\rho < \gamma_l$. By $\phi(\gamma_l, \langle p_{k'}^{\gamma_l} \mid k' < \lambda\rangle, C^{\gamma_l}, w\lceil\gamma_l)$ and $i \in S_1^2 \cap C^{\gamma_l}$, we have that $f(i) = i$, $p_i^{\gamma_k}\lceil\gamma_l = p_i^{\gamma_k 0}\lceil\gamma_l = p_i^{\gamma_l}$ and for all $j < i$, $p_j^{\gamma_k}\lceil\gamma_l = p_{f(j)}^{\gamma_k 0}\lceil\gamma_l = p_{f(j)}^{\gamma_l}$. Hence we have $p_i^{\gamma_k}\lceil\rho = p_i^{\gamma_l}\lceil\rho$ and $p_i^{\gamma_l}\lceil\rho$ forces the following;

$$p_i^{\gamma_l}(\rho) = \overline{\bigcup\{p_j^{\gamma_l}(\rho) \mid j < i\}}.$$

But $p_i^{\gamma_k}(\rho) = p_i^{\gamma_l}(\rho)$ and $\bigcup\{p_j^{\gamma_l}(\rho) \mid j < i\} = \bigcup\{p_{f(j)}^{\gamma_l}(\rho) \mid j < i\} = \bigcup\{p_j^{\gamma_k}(\rho) \mid j < i\}$, as $f(i) = i \in C(f)$. Hence we are done.

($i \in S_1^2 \cap C^{\gamma_k}$ and $\langle p_j^{\gamma_k} \mid j < i\rangle \equiv \langle q_{ij}\lceil\gamma_k \mid j < i\rangle$ implies $p_{i+1}^{\gamma_k} \le q_i^1\lceil\gamma_k$):

Let $i \in S_1^2 \cap C^{\gamma_k}$ and $\langle p_j^{\gamma_k} \mid j < i\rangle \equiv \langle q_{ij}\lceil\gamma_k \mid j < i\rangle$. Let $l < k$. It suffices to show $p_{i+1}^{\gamma_k}\lceil\gamma_l \le q_i^1\lceil\gamma_l$.

But $\langle p_j^{\gamma_l} \mid j < i\rangle \equiv \langle p_{f(j)}^{\gamma_l} \mid j < i\rangle = \langle p_{f(j)}^{\gamma_k 0}\lceil\gamma_l \mid j < i\rangle = \langle p_j^{\gamma_k}\lceil\gamma_l \mid j < i\rangle \equiv \langle q_{ij}\lceil\gamma_l \mid j < i\rangle$ and $i \in S_1^2 \cap C^{\gamma_l}$. Hence $p_{i+1}^{\gamma_k}\lceil\gamma_l = p_{f(i+1)}^{\gamma_l} \le p_{i+1}^{\gamma_l} \le q_i^1\lceil\gamma_l$ and $\gamma_l \le \alpha(i)$.

(For all $l < k$, we have $(\gamma_l, \langle p_i^{\gamma_l} \mid i < \lambda\rangle, C^{\gamma_l}, w\lceil\gamma_l)\, R\, (\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda\rangle, C^{\gamma_k}, w\lceil\gamma_k)$): For each $i < \lambda$, we have $p_i^{\gamma_k}\lceil\gamma_l = p_{f(i)}^{\gamma_k 0}\lceil\gamma_l = p_{f(i)}^{\gamma_l}$ and $f(i) \ge i$ holds.

Next let $i \in S_1^2 \cap C^{\gamma_k}$. Then $p_i^{\gamma_k} = p_{f(i)}^{\gamma_k 0} = p_i^{\gamma_k 0}$. Hence $p_i^{\gamma_k}\lceil\gamma_l = p_i^{\gamma_l}$.

Case. $\mathrm{cf}(\gamma) = \lambda = \omega_2^{V[G_\lambda]}$: Let $\eta < \gamma$ and $w \in P_\gamma$. We may assume, by increasing η, that $\mathrm{supp}(w) \subset \eta$. Let $\langle \gamma_k \mid k < \lambda \rangle$ be a sequence of ordinals which is continuously $<$-increasing, $\gamma_0 = \eta$, cofinal in γ and for each $i \in S_1^2$, we make sure that

$$\mathrm{supp}(q_i^1 \lceil \min\{\gamma, \alpha(i)\}) \subset \gamma_{i+1}.$$

Hence if $\gamma \leq \alpha(i)$, then $\mathrm{supp}(q_i^1 \lceil \gamma) \subset \gamma_{i+1}$. If $\alpha(i) < \gamma$, then $\mathrm{supp}(q_i^1) \subset \gamma_{i+1}$. This is possible as the supports are of size at most ω_1.

We construct $\langle p_i^{\gamma_k} \mid i < \lambda \rangle$ and C^{γ_k} by recusion on $k < \lambda$.

$\underline{k = 0}$: Let $\langle p_i^{\gamma_0} \mid i < \lambda \rangle = \langle p_i^{\eta} \mid i < \lambda \rangle$ and let $C^{\gamma_0} = C^{\eta}$. Then we have $\phi(\gamma_0, \langle p_i^{\gamma_0} \mid i < \lambda \rangle, C^{\gamma_0}, 1)$.

$\underline{k \text{ to } k+1}$: Suppose we have $\phi(\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda \rangle, C^{\gamma_k}, 1)$. Want $\phi(\gamma_{k+1}, \langle p_i^{\gamma_{k+1}} \mid i < \lambda \rangle, C^{\gamma_{k+1}}, 1)$ such that $(\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda \rangle, C^{\gamma_k}, 1) \, R \, (\gamma_{k+1}, \langle p_i^{\gamma_{k+1}} \mid i < \lambda \rangle, C^{\gamma_{k+1}}, 1)$. We just make sure to take care of the following situation. If $\mathrm{cf}(k) = \omega_1$, $\gamma \leq \alpha(k)$ and $p_{k+1}^{\gamma_k} \leq q_k^1 \lceil \gamma_k$, then consider

$$\phi(\gamma^k, \langle p_{k+1}^{\gamma_k} \mid 0 \leq i \leq k+1 \rangle ^\frown \langle p_i^{\gamma_k} \mid k+1 < i < \lambda \rangle, C^{\gamma_k} \cap (k+\omega, \lambda), w'),$$

where $w' = p_{k+1}^{\gamma_k} {}^\frown q_k^1 \lceil [\gamma_k, \gamma_{k+1}) \in P_{\gamma_{k+1}}$. Let $(\langle p_i^{\gamma_{k+1}} \mid i < \lambda \rangle, C^{\gamma_{k+1}})$ be such that

$$\phi(\gamma_{k+1}, \langle p_i^{\gamma_{k+1}} \mid i < \lambda \rangle, C^{\gamma_{k+1}}, w')$$

and

$$(\gamma_k, \langle p_{k+1}^{\gamma_k} \mid 0 \leq i \leq k+1 \rangle ^\frown \langle p_i^{\gamma_k} \mid k+1 < i < \lambda \rangle, C^{\gamma_k} \cap (k+\omega, \lambda), p_{k+1}^{\gamma_k}) \, R \, (\gamma_{k+1}, \langle p_i^{\gamma_{k+1}} \mid i < \lambda \rangle, C^{\gamma_{k+1}}, w').$$

We have that

$$p_0^{\gamma_{k+1}} \leq q_k^1 \lceil \gamma_{k+1}$$

and that

$$(\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda \rangle, C^{\gamma_k}, 1) \, R \, (\gamma_{k+1}, \langle p_i^{\gamma_{k+1}} \mid i < \lambda \rangle, C^{\gamma_{k+1}}, 1).$$

$\underline{k \text{ limit}}$: We have $\mathrm{cf}(k) < \lambda = \omega_2^{V[G_\lambda]}$. Hence there exists $\langle p_i^{\gamma_k} \mid i < \lambda \rangle$ and C^{γ_k} such that $\phi(\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda \rangle, C^{\gamma_k}, 1)$ and that for all $l < k$, $(\gamma_l, \langle p_i^{\gamma_l} \mid i < \lambda \rangle, C^{\gamma_l}, 1) \, R \, (\gamma_k, \langle p_i^{\gamma_k} \mid i < \lambda \rangle, C^{\gamma_k}, 1)$.

This completes the construction of $(\langle p_i^{\gamma_k} \mid i < \lambda \rangle, C^{\gamma_k})$. Now we begin a sort of diagonal construction. Let

$$C^{\gamma 0} = \{k < \lambda \mid k \in C_{\gamma_l \gamma_m} \text{ for all } l < m < k\}.$$

For each $i \in S_1^2 \cap C^{\gamma 0}$, let

$$p_i^{\gamma 0} = (\bigcup \{p_i^{\gamma_l} \mid l < i\}) ^\frown 1 \in P_\gamma.$$

Let $f : \lambda \longrightarrow S_1^2 \cap C^{\gamma 0}$ be the $\in$-isomorphism. Let

$$C^{\gamma} = C^{\gamma 0} \cap C(f)$$

and for each $i < \lambda$, let

$$p_i^{\gamma} = p_{f(i)}^{\gamma 0}.$$

Want $\phi(\gamma, \langle p_i^{\gamma} \mid i < \lambda \rangle, C^{\gamma}, w)$ and that $(\eta, \langle p_i^{\eta} \mid i < \lambda \rangle, C^{\eta}, w \lceil \eta) \, R \, (\gamma, \langle p_i^{\gamma} \mid i < \lambda \rangle, C^{\gamma}, w)$.

Some details.

(p_i^{γ} is descending): $j < i$ implies $f(j) < f(i)$. Hence for any $l < f(j)$, we have $p_i^{\gamma} \lceil \gamma_l = p_{f(i)}^{\gamma_l} \leq p_{f(j)}^{\gamma_l} = p_j^{\gamma} \lceil \gamma_l$. Hence $p_i^{\gamma} \lceil \gamma_{f(j)} \leq p_j^{\gamma} \lceil \gamma_{f(j)}$ and so $p_i^{\gamma} \leq p_j^{\gamma}$.

($i \in S^2_1 \cap C^\gamma$ implies $p^\gamma_i \equiv \langle p^\gamma_j \mid j < i \rangle$): Let $\rho < \gamma$. We first assume that $\rho < \gamma_i$. Then for any $l < i$ such that $\rho < \gamma_l$, we have $p^\gamma_i \lceil \rho = p^{\gamma_l}_i \lceil \rho$ and $p^{\gamma_l}_i \lceil \rho$ forces the following;

$$p^{\gamma_l}_i(\rho) = \overline{\bigcup \{p^{\gamma_l}_j(\rho) \mid j < i\}}.$$

But $p^\gamma_i(\rho) = p^{\gamma_l}_i(\rho)$ and $\langle p^\gamma_j(\rho) \mid j < i \rangle \equiv \langle p^{\gamma_l}_{f(j)}(\rho) \mid l < f(j), j < i \rangle \equiv \langle p^{\gamma_l}_j(\rho) \mid j < i \rangle$, as $f(i) = i \in C(f)$. Hence $p^\gamma_i \lceil \rho$ forces the following;

$$p^\gamma_i(\rho) = \overline{\bigcup \{p^\gamma_j(\rho) \mid j < i\}}.$$

We next assume $\gamma_i \leq \rho$. Then for all $j \leq i$, we have $p^\gamma_j(\rho) = \emptyset$.

($i \in S^2_1 \cap C^\gamma$ and $\langle p^\gamma_j \mid j < i \rangle \equiv \langle q_{ij} \lceil \gamma \mid j < i \rangle$ implies $p^\gamma_{i+1} \leq q^1_i \lceil \gamma$): Let $i \in S^2_1 \cap C^\gamma$ and $\langle p^\gamma_j \mid j < i \rangle \equiv \langle q_{ij} \lceil \gamma \mid j < i \rangle$. Let l be any with $l < i$. Then $\langle p^{\gamma_l}_j \mid j < i \rangle \equiv \langle p^{\gamma_l}_{f(j)} \mid j < i \rangle \equiv \langle p^\gamma_j \lceil \gamma_l \mid j < i \rangle \equiv \langle q_{ij} \lceil \gamma_l \mid j < i \rangle$. Since $i \in S^2_1 \cap C^{\gamma_l}$, we have $p^{\gamma_l}_{i+1} \leq q^1_i \lceil \gamma_l$. Hence $p^{\gamma_i}_{i+1} \lceil \gamma_l = p^{\gamma_l}_m \leq p^{\gamma_l}_{i+1} \leq q^1_i \lceil \gamma_l$ for some $m \geq i+1$. Hence we conclude $p^{\gamma_i}_{i+1} \leq q^1_i \lceil \gamma_i$ and $\gamma \leq \alpha(i)$. By construction, we have $i+1 < f(i+1) \in S^2_1$ and so $p^\gamma_{i+1} \lceil \gamma_{i+1} = p^{\gamma_{i+1}}_{f(i+1)} \leq p^{\gamma_{i+1}}_0 \leq q^1_i \lceil \gamma_{i+1}$. But $\mathrm{supp}(q^1_i \lceil \gamma) \subset \gamma_{i+1}$. Hence $p^\gamma_{i+1} \leq q^1_i \lceil \gamma$.

($(\eta, \langle p^\eta_i \mid i < \lambda, C^\eta, w \lceil \eta) \, R \, (\gamma, \langle p^\gamma_i \mid i < \lambda, C^\gamma, w)$): $C^\eta \supset C^\gamma$.
$p^\gamma_i \lceil \eta = p^\gamma_i \lceil \gamma_0 = p^{\gamma_0}_{f(i)}$ and $i \leq f(i)$.
For $i \in S^2_1 \cap C^\gamma$, we have $p^\gamma_i \lceil \eta = p^\eta_i$, as $f(i) = i$.

$\square$

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